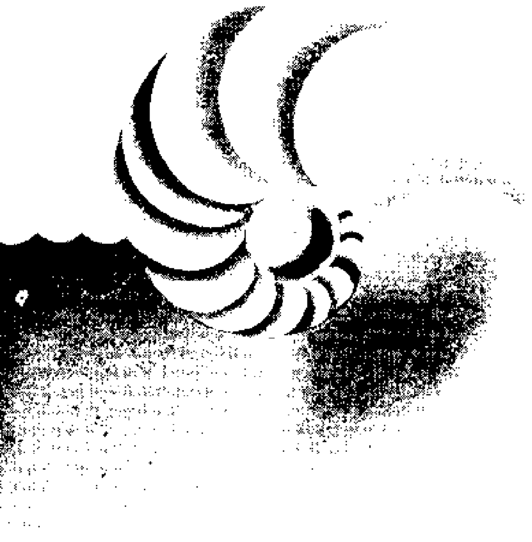

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TECHNICAL REPORT

Sound Propagation in a Liquid Layer Overlying A Multi-layered Viscoelastic Halfspace

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A Report of a
Cooperative University-Industry Research Project
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SOUND PROPAGATION IN A LIQUID LAYER OVERLYING A MULTI-LAYERED VISCOELASTIC HALFSPACE

by

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ABSTRACT

The acoustic response due to a point source in a liquid layer overlying a semi-infinite multi-layered viscoelastic medium is obtained using a Green's function formalism. A matrix recurrence relation, developed from the boundary conditions, is used to relate the scalar wave functions in the last viscoelastic layer to the scalar wave functions in the intermediate layers using (4x4) matrix manipulations. The transformed form of the Green's function is then obtained by applying appropriate boundary conditions at the top and bottom of the liquid layer. The Green's function is then written in integral form convenient for computer evaluation, since the integrand can be computer for many layers using the recurrence relation. Special cases are discussed and compared with known results.

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Abstract

The acoustic response due to a point source in a liquid layer overlying a semi-infinite multi-layered viscoelastic medium is obtained using a Green's function formalism. A matrix recurrence relation, developed from the boundary conditions, is used to relate the scalar wave functions in the last viscoelastic layer to the scalar wave functions in the intermediate layers using (4×4) matrix manipulations. The transformed form of the Green's function is then obtained by applying appropriate boundary conditions at the top and bottom of the liquid layer. The Green's function is then written in integral form convenient for computer evaluation, since the integrand can be computed for many layers using the recurrence relation. Special cases are discussed and compared with known results.

Introduction

The first paper of this series treated the acoustic response of a semi-infinite liquid overlying a homogeneous viscoelastic halfspace (Ref. 1). This paper treats the more general case of a point source in a liquid layer of finite depth overlying a series of n parallel layers of a viscoelastic solid (Fig. 1). Each layer has arbitrary density ρ_j, and complex velocities of wave propagation α_j and β_j. This model corresponds rather closely to the stratification of sediments on the continental shelf. Each subbottom layer is assumed to be a homogeneous, isotropic Voigt solid, described mathematically in the main text of this paper.

This problem and closely related problems have been studied by numerous investigators(2),(3),(4),(5). Jardetsky(2), developed the period equation or dispersion relation for a layered elastic halfspace with a point source in the first layer. The period equation was expressed as a determinant of order (4n-2), where n was the number of elastic layers. He could not obtain the roots of the period equation explicitly due to the algebraic complexity, but he did make an important observation. This was that, of all the 2n branch point singularities, only the two branch line integrals corresponding to the two branch points of the last (semi-infinite) layer contribute to the response. Thus, the total response consists of a residue series (each term of which corresponds to a solution of the period equation) and contributions due to branch line integrals for the branch point singularities of the last (semi-infinite) layer.

Thomson(3) used a matrix formalism for determining the transmission of plane elastic waves through a stratified solid medium. He developed a recurrence relation relating the velocities and stresses in adjoining plates or layers using Snell's law and continuity of particle velocities and stresses at the interface. Successive application of the recurrence relation enabled him to relate the velocities and stresses at the last plate in terms of those of the first plate. Shaw and Bugl(4) pointed out that Thomson used the unnecessarily restrictive assumption that the shear modulus was constant in all the layers. They also mention that Haskell(5) was apparently the first to remove this restriction.

Shaw and Bugl(4) refined the approach of Refs. (3) and (5) by expressing the displacements and stresses in terms of the layer's parameters and then used a more direct matrix formalism. In addition, they considered the effects of viscoelasticity by treating the elastic constants λ and μ as complex quantities.

The present paper closely parallels the approach of Ref. (2), except that the first layer is taken to be a liquid. This enables us to express the response due to the point source using a scalar Green's function formalism(6), (7). We depart from Ref. (2) to develop a recurrence relation between the coefficients of the scalar potentials of adjacent viscoelastic layers by applying boundary conditions at the interface. Successive application of the recurrence relation then enables us to express the coefficients of the first solid layer in terms of the last layer. Finally, boundary conditions are applied at the top and bottom of the liquid layer to obtain the solution for the response in the liquid. The advantage of this approach is that computations involve 4×4 matrices instead of matrices of order (4n-2), where n is the number of solid layers. This makes the formalism ideal for machine computation.

The recurrence relation developed here differs from that of Refs. (3), (4) and (5) in that the problem treated is three-dimensional and the recurrence relation involves coefficients of scalar potentials instead of displacements and stresses.

Analytical Development

The governing equation in the liquid with a point source at $r = 0$, $z = z'$ [Fig. (1)] is taken from Eq. (A-2) of the preceding paper[1].

$$(\nabla^2 + k_0^2)G(\vec{r}|\vec{r}', \omega) = -\delta(\vec{r}-\vec{r}') \tag{1}$$

where $\delta(\vec{r}-\vec{r}') = \dfrac{\delta(r)\delta(z-z')}{2\pi r}$,

$$G(\vec{r}|\vec{r}', \omega) = G(r,z,z',\omega),$$

and $k_0^2 = \omega^2/c_0^2$.

The quantities G and c_0 are the Green's function and the liquid's speed of sound (adiabatic), respectively. The symbol δ is Dirac's delta function. A time dependence of the form $e^{i\omega t}$ is taken throughout, and is omitted for brevity.

The governing equations for the $j^{\underline{th}}$ viscoelastic layer may be written

$$\rho_j\omega^2\vec{u}_j + \mu_j\nabla^2\vec{u}_j + (\lambda_j+\mu_j)\nabla(\nabla\cdot\vec{u}_j) = 0 \tag{2}$$

where ρ_j, $\vec{u}_j$, μ_j, λ_j are the layer's density, displacement and Lamé parameters. We introduce Voigt viscoelasticity by writing the Lamé parameters in the frequency domain as

$$\mu = \mu' + i\omega\mu''$$

and

$$\lambda = \lambda' + i\omega\lambda'' \quad .$$

Following Ref. (1), Eq. (2) reduces to two scalar Helmholtz equations:

$$(\nabla^2 + k_{\alpha j}^2)\phi_{\alpha j} = 0 \tag{2-a}$$

and

$$(\nabla^2 + k_{\beta j}^2)\phi_{\beta j} = 0 \ . \tag{2-b}$$

where $k_{\alpha j}^2 = \dfrac{\omega^2}{\alpha_j^2}$, $k_{\beta j}^2 = \dfrac{\omega^2}{\beta_j^2}$,

the speed of the dilatational wave α_j and the shear wave β_j are given by

$$\alpha_j^2 = \frac{\lambda_j'+2\mu_j'+i\omega(\lambda_j''+2\mu'')}{\rho_j} ,$$

and

$$\beta_j^2 = \frac{\mu_j'+i\omega\mu_j''}{\rho_j} \ .$$

The displacements may be written as

$$\vec{u}_j = \vec{u}_{\alpha j} + \vec{u}_{\beta j} , \tag{3}$$

where $\vec{u}_{\alpha j} = \nabla\phi_{\alpha j}$,

$$\vec{u}_{\beta j} = \nabla\times\nabla\times(\hat{e}_z\phi_{\beta j}) ,$$

$$\nabla\cdot\vec{u}_{\beta j} = 0, \quad \nabla\times\vec{u}_{\alpha j} = 0 ,$$

and $\hat{e}_z$ is the unit vector in the z-direction.

Solutions to Eqs. (1) and (2) may be obtained conveniently using a Fourier-Bessel transform

$$\underline{A}(\zeta) = \int_0^\infty A(r)J_0(\zeta r)rdr \tag{4-a}$$

and the inverse transform

$$A(r) = \int_0^\infty \underline{A}(\zeta)J_0(\zeta r)\zeta d\zeta \ . \tag{4-b}$$

The lower bar denotes a transformed quantity. Applying the transformation (4-a) to Eq. (1) results in the following differential equation

$$\left[\frac{d^2}{dz^2} - a_0^2\right] \underline{G}(\zeta,z,z') = \frac{-\delta(z-z')}{2\pi} \tag{5}$$

where $a_0 = \sqrt{\zeta^2 - k_0^2}$.

We write solutions to Eq. (5) above and below the source as

$$\underline{G}_> = Pe^{-a_0 z} + Qe^{a_0 z} , \quad z' < z < h_0 \tag{6-a}$$

$$\underline{G}_< = Re^{-a_0 z} + Se^{a_0 z} , \quad 0 < z < z' . \tag{6-b}$$

We may eliminate three of the unknowns in Eqs. (6) using the continuity and jump condition at z=z' (7) and by noting that the pressure, which is proportional to G, vanishes at $z=h_0$. Eqs. (6-a and b) may then be written as

$$\underline{G}_< = \frac{1}{4\pi a_0}\left\{A_0\sinh[a_0(h_0-z)] - 2\sinh[a_0(z'-z)]\right\}, \tag{6-c}$$

and

$$\underline{G}_> = \frac{A_0}{4\pi a_0}\sinh[a_0(h_0-z)] , \tag{6-d}$$

where A_0 is an unknown, as yet, function of ζ to be determined from the boundary conditions at z=0.

Solutions to Eqs. (2-a and b) may be written in transformed form as

$$\underline{\phi}_{\alpha j} = \frac{1}{4\pi a_{\alpha j}}\left\{A_j e^{a_{\alpha j}z} + B_j e^{-a_{\alpha j}z}\right\}, \tag{7-a}$$

and

$$\underline{\phi}_{\beta j} = \frac{1}{4\pi a_{\beta j}}\left\{C_j e^{a_{\beta j}z} + D_j e^{-a_{\beta j}z}\right\}, \tag{7-b}$$

where $a_{\alpha j} = \sqrt{\zeta^2 - k_{\alpha j}^2}$,

$$a_{\beta j} = \sqrt{\zeta^2 - k_{\beta j}^2} \quad ,$$

and $h_{(j-1)} < z < h_j$. We note that the n^{th} semi-infinite layer has only two terms, as $B_n = 0$, $D_n = 0$, because the potentials must remain finite as $z \to -\infty$. This implies that the A_j and C_j terms in Eqs. (7) represent downward-traveling waves and the B_j and D_j terms represent upward-traveling waves. It follows that the stipulation that $B_n = D_n = 0$ is essentially a radiation condition.

Boundary Conditions

We must evaluate the 4n-2 functions A_j, B_j, C_j, D_j, $j = 1, 2, \ldots n$ in Eq. (7). We do this by applying boundary conditions at each interface between solid layers. We first express the displacements and stresses in terms of the potentials. The displacements in a cylindrical (r, z, θ) coordinate system may be written (taking into account the θ-symmetry) as

$$u_r = \frac{\partial}{\partial r} \left(\phi_\alpha + \frac{\partial \phi_\beta}{\partial z}\right) \quad , \tag{8-a}$$

$$u_z = \frac{\partial \phi_\alpha}{\partial z} + \left(k_\beta^2 + \frac{\partial^2}{\partial z^2}\right)\phi_\beta \quad , \tag{8-b}$$

and

$$u_\theta = 0 \quad . \tag{8-c}$$

We write the stress tensor as

$$\sigma_{ik} = \lambda \varepsilon_{\ell\ell} \delta_{ik} + 2\mu \varepsilon_{ik} \quad , \tag{8-d}$$

where ε_{ik} is the strain tensor and δ_{ik} is the Kronecker delta. We need the stress components σ_{zz} and σ_{zr}, whose corresponding strains are written as

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{8-e}$$

and

$$2\varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \quad . \tag{8-f}$$

The boundary conditions are continuity of stress and displacements at the interface. The first follows from the dynamic equations of motion. The second follows from the conservation of mass. For the j^{th} interface, we write

$$\text{i)} \quad u_{zj} = u_{z(j+1)} \quad \text{at } z = -h_j \quad , \tag{9-a}$$

$$\text{ii)} \quad u_{rj} = u_{r(j+1)} \quad '' \quad , \tag{9-b}$$

$$\text{iii)} \quad \sigma_{rzj} = \sigma_{rz(j+1)} \quad '' \quad , \tag{9-c}$$

$$\text{iv)} \quad \sigma_{zzj} = \sigma_{zz(j+1)} \quad '' \quad . \tag{9-d}$$

Applying Eqs. (8) to (9) gives a system of equations that may be written in matrix form as

$$[a_j]\, \vec{A}_{j,j} = [a_{(j+1)}]\, \vec{A}_{(j+1),j} \quad , \tag{10}$$

where $[a_j]$ is a 4×4 matrix given as:

$$[a_j] = \begin{bmatrix} \rho_j \beta_j^2 (2\zeta^2 - k_{\beta j}^2) & \rho_j \beta_j^2 (2\zeta^2 - k_{\beta j}^2) & -2\rho_j \beta_j^2 a_{\beta j} \zeta^2 & 2\rho_j \beta_j^2 a_{\beta j} \zeta^2 \\ -2\rho_j \beta_j^2 a_{\alpha j} & 2\rho_j \beta_j^2 a_{\alpha j} & \rho_j \beta_j^2 (2\zeta^2 - k_{\beta j}^2) & \rho_j \beta_j^2 (2\zeta^2 - k_{\beta j}^2) \\ 1 & 1 & -a_{\beta j} & a_{\beta j} \\ -a_{\alpha j} & a_{\alpha j} & \zeta^2 & \zeta^2 \end{bmatrix} \tag{11-a}$$

and $\vec{A}_{(j+1),j}$ is a (4×1) column matrix (vector), setting $(j+1) = j'$:

$$\vec{A}_{j',j} = \begin{bmatrix} \dfrac{A_{j'} e^{-a_{\alpha j'} h_j}}{a_{\alpha j'}} \\ \dfrac{B_{j'} e^{+a_{\alpha j'} h_j}}{a_{\alpha j'}} \\ \dfrac{C_{j'} e^{-a_{\beta j'} h_j}}{a_{\beta j'}} \\ \dfrac{D_{j'} e^{a_{\beta j'} h_j}}{a_{\beta j'}} \end{bmatrix} \tag{11-b}$$

One may express $\vec{A}_{j',j}$ as $\vec{A}_{j',j} = [A_{j',j}]\vec{A}_j$

where

$$[A_{j',j}] = \begin{bmatrix} e^{-a_{\alpha j'}h_j} & 0 & 0 & 0 \\ 0 & e^{a_{\alpha j'}h_j} & 0 & 0 \\ 0 & 0 & e^{-a_{\beta j'}h_j} & 0 \\ 0 & 0 & 0 & e^{a_{\beta j'}h_j} \end{bmatrix} \qquad (11\text{-c})$$

and

$$\vec{A}_{j'} = \begin{bmatrix} \frac{A_{j'}}{a_{\alpha j'}} \\ \frac{B_{j'}}{a_{\alpha j'}} \\ \frac{C_{j'}}{a_{\beta j'}} \\ \frac{D_{j'}}{a_{\beta j'}} \end{bmatrix} . \qquad (11\text{-d})$$

We now may write Eq. (10) as

$$[a_j][A_{j,j}]\vec{A}_j = [a_{j'}][A_{j',j}]\vec{A}_{j'} \quad . \qquad (12)$$

Eq. (12) is a recurrence relation relating the coefficients of the j^{th} layer to the $j'=(j+1)^{th}$ layer. Solving for A_j gives

$$\vec{A}_j = [A_{j,j}]^{-1}[a_j]^{-1}[a_{j'}][A_{j',j}]\vec{A}_{j'} \quad . \qquad (12\text{-a})$$

We may define

$$[b_{j',j}] = [A_{j,j}]^{-1}[a_j]^{-1}[a_{j'}][A_{j',j}], \qquad (12\text{-b})$$

which allows Eq. (12) to be written in a simpler form:

$$\vec{A}_j = [b_{(j+1),j}]\, \vec{A}_{(j+1)} \ . \qquad (12\text{-c})$$

One may apply the recurrence relation (12-c) successively to eliminate the coefficients of intermediate layers. In particular, one may express the coefficients of the first layer (j=1) to those of the last layer (j=n) as a product

$$\vec{A}_1 = \left\{ \prod_{\ell=1}^{n-1} [b_{(\ell+1),\ell}] \right\} \vec{A}_n \quad , \qquad (13)$$

where we may set

$$[M] = \prod_{\ell=1}^{n-1} [b_{(\ell+1),\ell}] \quad , \qquad (13\text{-a})$$

where [M] is a (4×4) matrix, so

$$\vec{A}_1 = [M]\ \vec{A}_n \ . \qquad (13\text{-b})$$

We recall that the second and fourth elements of $\vec{A}_n$ are zero due to a radiation condition. If one denotes the elements of [M] as m_{ik}, one may expand Eq. (13-b) as follows:

$$\begin{aligned} \frac{A_1}{a_{\alpha 1}} &= m_{11}\frac{A_n}{a_{\alpha n}} + m_{13}\frac{C_n}{a_{\beta n}} \\ \frac{B_1}{a_{\alpha 1}} &= m_{21}\frac{A_n}{a_{\alpha n}} + m_{23}\frac{C_n}{a_{\beta n}} \\ \frac{C_1}{a_{\beta 1}} &= m_{31}\frac{A_n}{a_{\alpha n}} + m_{33}\frac{C_n}{a_{\beta n}} \\ \frac{D_1}{a_{\beta 1}} &= m_{41}\frac{A_n}{a_{\alpha n}} + m_{43}\frac{C_n}{a_{\beta n}} \ . \end{aligned} \qquad (13\text{-c})$$

Eq. (13-c) relates the four coefficients of the first solid layer to the two coefficients of the last solid layer by means of the elements of the [M] matrix. The elements of the [M] matrix can be computed as a function of ζ knowing the densities and wave numbers of the viscoelastic layers.

Now one may relate the four coefficients of the first solid layer to the unknown coefficient A_0 [Eq. (6)] of the liquid layer by applying boundary conditions at the liquid-solid interface z=0. We write

$$\text{i)}\quad \sigma_{zz_0} = -p_0 = \sigma_{zz_1} \quad \text{at } z = 0 \ , \qquad (14\text{-a})$$

$$\text{ii)}\quad u_{z_0} = u_{z_1} \qquad \text{"} \qquad (14\text{-b})$$

$$\text{and}\quad \text{iii)}\quad \sigma_{rz_1} = 0 \qquad \text{"} \quad . \qquad (14\text{-c})$$

where p_0 is the liquid pressure given as $p_0 = -\rho_0\omega^2 G$ for $\vec{r} \neq \vec{r}'$. Using Eqs. (8) in (14) results in the matrix relation:

$$\begin{bmatrix} a_0\cosh\ a_0h_0 & a_{\alpha 1} & -a_{\alpha 1} & -\zeta^2 & -\zeta^2 \\ \rho_0\omega^2\sinh\ a_0h_0 & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & -2\rho_1\beta_1^2\zeta^2a_{\beta 1} & 2\rho_1\beta_1^2\zeta^2a_{\beta 1} \\ 0 & -2a_{\alpha 1} & 2a_{\alpha 1} & (2\zeta^2-k_{\beta 1}^2) & (2\zeta^2-k_{\beta 1}^2) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{A_0}{a_0} \\ \frac{A_1}{a_{\alpha 1}} \\ \frac{B_1}{a_{\alpha 1}} \\ \frac{C_1}{a_{\beta 1}} \\ \frac{D_1}{a_{\beta 1}} \end{bmatrix} = \frac{2}{a_0} \begin{bmatrix} a_0\cosh\ a_0z' \\ \rho_0\omega^2\sinh\ a_0z' \\ 0 \end{bmatrix} . \qquad (15)$$

Applying the result (13-c) to Eq. (15) gives

$$\begin{bmatrix} a_0\cosh\ a_0h_0 & b_{12} & b_{13} \\ \rho_0\omega^2\sinh\ a_0h_0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} A_0/a_0 \\ A_n/a_{\alpha n} \\ C_n/a_{\alpha n} \end{bmatrix} = \frac{2}{a_0} \begin{bmatrix} a_0\cosh\ a_0z' \\ \rho_0\omega^2\sinh\ a_0z' \\ 0 \end{bmatrix} , \qquad (16)$$

where

$$b_{11} = (m_{11}-m_{21})a_{\alpha 1} - \zeta^2(m_{31}+m_{41}) ,$$

$$b_{22} = \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2)(m_{11}+m_{21}) - - 2\rho_1\beta_1^2\zeta^2a_{\beta 1}(m_{31}-m_{41}) ,$$

$$b_{32} = -2a_{\alpha 1}(m_{11}-m_{21}) + (2\zeta^2-k_{\beta 1}^2)(m_{31}+m_{41}) ,$$

$$b_{13} = (m_{13}-m_{23})a_{\alpha 1} - \zeta^2(m_{33}+m_{43}) ,$$

$$b_{23} = \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2)(m_{13}+m_{23}) - - 2\rho_1\beta_1^2a_{\beta 1}(m_{33}-m_{43})$$

and $b_{33} = -2a_{\alpha 1}(m_{13}-m_{23}) + (2\zeta^2-k_{\beta 1}^2)(m_{33}+m_{43})$.

We may solve for A_0 from Eq. (16) using Cramer's rule

$$A_0 = \frac{2}{a_0}\frac{\Delta_1}{\Delta_0} , \qquad (17)$$

where

$$\Delta_1 = K_1a_0\ \cosh(a_0z') - K_2\rho_0\omega^2\sinh(a_0z'), \qquad (17\text{-a})$$

$$\Delta_0 = K_1a_0\ \cosh(a_0h_0) - K_2\rho_0\omega^2\sinh(a_0h_0), \qquad (17\text{-b})$$

$$K_1 = b_{22}b_{33} - b_{32}b_{23} ,$$

and

$$K_2 = b_{12}b_{33} - b_{13}b_{32} .$$

Substituting the result (17) into the expression for the Green's function [Eq. (6)] yields

$$\underline{G}(\zeta,z,z') = \frac{2}{4\pi a_0}\sinh[a_0(h_0+z_>)] \times \times\left\{\frac{K_1a_0\cosh(a_0z_<) - K_2\rho_0\omega^2\sinh(a_0z_<)}{K_1a_0\cosh(a_0h_0) - K_2\rho_0\omega^2\sinh(a_0h_0)}\right\} , \qquad (18)$$

where $z_> = \mathrm{Max}(z,z')$ and $z_< = \mathrm{Min}(z,z')$. Using the symbolism $z_>$, $z_<$ combines the two expressions Eqs. (6-a and b) for $\underline{G}_>$ and $\underline{G}_<$ into one due to reciprocity. We note that the period equation is obtained by setting the denominator of Eq. (18) to zero, or from Eq. (17-b)

$$\Delta_0 = 0 .$$

The actual Green's function $G(r,z,z',\omega)$ is obtained by taking the inverse transform of Eq. (18) using Eq. (4-b)

$$G(r,z,z',\omega) = \int_0^\infty \underline{G}(\zeta,z,z',\omega)\ \ J_0(\zeta r)\zeta d\zeta . \qquad (18\text{-a})$$

Special Cases

1) One viscoelastic layer (n=1). Here we set in Eq. (13-c) $m_{11} = m_{33} = 1$ and the other $m_{ij} = 0$. The Green's function reduces to $\underline{G}(\zeta,z,z')=$

$$= \frac{2}{4\pi a_0}\sinh[a_0(h_0+z_>)]\left\{\frac{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 z_<)-a_{\alpha 1}k_{\beta 1}^4\sinh(a_0 z_<)}{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 h_0)-a_{\alpha 1}k_{\beta 1}^4\sinh(a_0 h_0)}\right\} . \tag{19}$$

Eq. (19) agrees, after taking the inverse transform, with Press and Ewing's[8] result for the liquid layer over a semi-infinite elastic solid [their Eqs. (26) and (27)]. Our result includes an extra $1/4\pi$ factor that results from the Green's function formalism, and the notation and sign conventions differ.

2) Infinite depth of liquid layer ($h_0\to\infty$). In this case Eq. (18) reduces to:

$$\underline{G}(\zeta,z,z') = \frac{2}{4\pi a_0}e^{-(a_0 z_>)}\left\{\frac{K_1 a_0\cosh(a_0 z_<)-K_2\rho_0\omega^2\sinh(a_0 z_<)}{(K_1 a_0 - K_2\rho_0\omega^2)}\right\} \tag{20}$$

Here the frequency equation is simply

$$K_1 a_0 - K_2\rho_0\omega^2 = 0 \quad . \tag{20-a}$$

3) Semi-infinite liquid over a viscoelastic halfspace ($h_0\to\infty$, n=1). The Green's function reduces to:

$$\underline{G}(\zeta,z,z') = \frac{2}{4\pi a_0}e^{-(a_0 z_>)}\left\{\frac{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2 - 4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 z_<) + k_{\beta 1}^4 a_{\alpha 1}\sinh(a_0 z_<)}{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2 - 4a_{\alpha 1}a_{\beta 1}\zeta^2] + k_{\beta 1}^4 a_{\alpha 1}}\right\} . \tag{21}$$

This result is the same as Eq. (A-7) of the preceding paper[1], as would be expected.

Results and Conclusions

A general expression [Eq. (18) is obtained, in transformed form, for the acoustic response due to a point source in a liquid layer overlying a multi-layered viscoelastic solid halfspace. Special cases of the result are presented [Eqs. (19), (20) and (21)] for one viscoelastic layer, infinite liquid depth and one viscoelastic layer combined with the infinite liquid depth.

To obtain the actual response, one must take the inverse transform of the expression [Eq. (18), or one of its special cases]. Taking the inverse transform requires evaluating a definite integral of the form indicated in Eq. (18-a). This integration was discussed[1] and approximate results (leading terms in an asymptotic expansion[9]) were obtained for the special case corresponding to Eq. (21).

The same techniques may be applied to Eq. (20) (the infinite liquid depth case), as the integrand is in the same form. The period equation [Eq. (20-a)] becomes more complicated due to the presence of solid layers between the two halfspaces. The solid layers (plates) produce a waveguide-like effect which manifests itself in a residue series, each term representing one mode of propagation. As pointed out by Jardetsky[2], only two branch line integrals will contribute to the response. These integrals correspond to the branch point singularities of the last viscoelastic layer ($a_{\alpha n} = 0$, $a_{\beta n} = 0$).

In the radiation zone, the steady-state plane-wave reflection coefficient may be obtained from Eq. (20) by expanding the sinh and cosh terms to yield the following:

$$\underline{G}(\zeta,z,z') = \frac{1}{4\pi a_0}\left[e^{-a_0(z_>-z_<)} + e^{-a_0(z_>+z_<)}\times\left\{\frac{K_1 a_0 + K_2\rho_0\omega^2}{K_1 a_0 - K_2\rho_0\omega^2}\right\}\right] . \tag{22}$$

The first term in Eq. (22) is the direct wave, as may be seen from Sommerfeld's[10] result. The second term represents the reflected wave. This follows from the leading term of a steepest descent integration of the inverse transform of the second term.[1] Following the development of Ref. (1) gives, for the plane-wave reflection coefficient:

$$\left[\frac{K_1 a_0 + K_2\rho_0\omega^2}{K_1 a_0 - K_2\rho_0\omega^2}\right]_{\zeta=\zeta_0} \tag{22-a}$$

where $\zeta_0 = k_0\sin\theta$, $\theta = \sin^{-1}(\frac{r}{R_I})$ is the angle of incidence, and $R_I = \sqrt{r^2+(z+z')^2}$. One notes that the reflection coefficient given in Eq. (22-a) is in a form similar to the usual impedance relation representing the reflection coefficient for two liquids

$$\frac{Z_1 - Z_0}{Z_1 + Z_0}$$

where $Z_0 = \rho_0 c_0$, $Z_1 = \rho_1 \alpha_1$.

For reasons discussed in Ref. (1), the primary interest in our research is in the special case corresponding to infinite water depth. The explicit calculation of the acoustic response for the two-layer case (n=2) using the methods of Ref. (1) is an ambitious undertaking due to the algebraic complexity. For three layers or more, direct calculations become unmanageable. For this reason, further development will be accomplished using machine calculations. That is, the integrand will be evaluated for arbitrary layers with the aid of the recurrence relation, and the subsequent integration will be done numerically. In addition, studies will be done separately to find the roots of the period equation [Eq. (20-a)].

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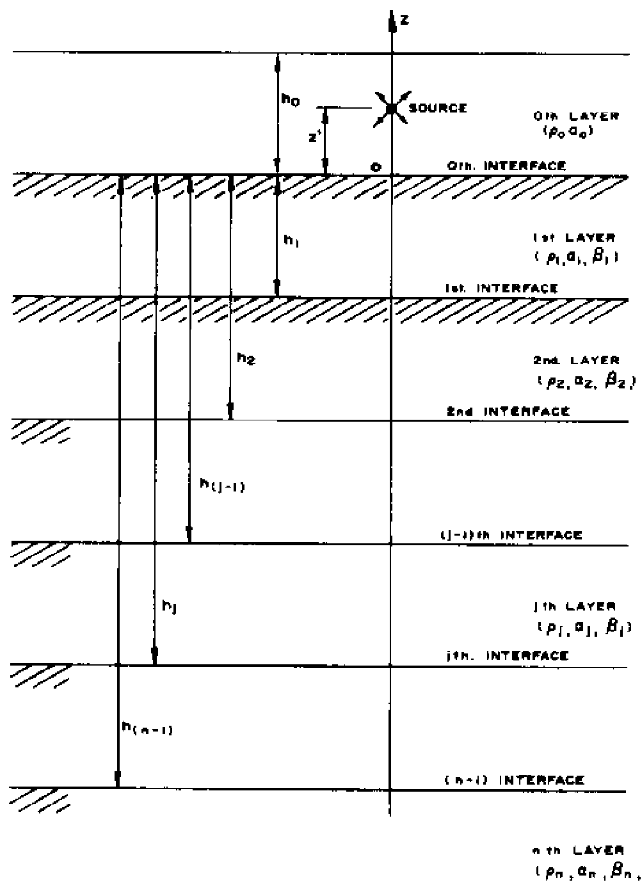

Figure 1. Dimensions and Coordinate System Used for Multi-layer Problem

ERRATA

"Sound Propagation in a Liquid Layer Overlying a Multi-Layered Viscoelastic Halfspace"

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equation (11-a)

$$[a_j] = \begin{bmatrix} \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & 2\rho_j\beta_j^2 a_{\beta j}\zeta^2 & -2\rho_j\beta_j^2 a_{\beta j}\zeta^2 \\ 2\rho_j\beta_j^2 a_{\alpha j} & -2\rho_j\beta_j^2 a_{\alpha j} & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) \\ 1 & 1 & a_{\beta j} & -a_{\beta j} \\ a_{\alpha j} & -a_{\alpha j} & \zeta^2 & \zeta^2 \end{bmatrix}$$

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equation (15)

$$\begin{bmatrix} a_0\cosh a_0h_0 & a_{\alpha 1} & -a_{\alpha 1} & \zeta^2 & \zeta^2 \\ \rho_0\omega^2\sinh a_0h_0 & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & 2\rho_1\beta_1^2\zeta^2 a_{\beta 1} & -2\rho_1\beta_1^2\zeta^2 a_{\beta 1} \\ 0 & 2a_{\alpha 1} & -2a_{\alpha 1} & (2\zeta^2-k_{\beta 1}^2) & (2\zeta^2-k_{\beta 1}^2) \end{bmatrix} \times \cdots$$

$$b_{12} = (m_{11}-m_{21})a_{\alpha 1} + \zeta^2(m_{31}+m_{41})$$

$$b_{22} = \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2)(m_{11}+m_{21}) + 2\rho_1\beta_1^2\zeta^2 a_{\beta 1}(m_{31}-m_{41})$$

$$b_{32} = 2a_{\alpha 1}(m_{11}-m_{21}) + (2\zeta^2-k_{\beta 1}^2)(m_{31}+m_{41})$$

$$b_{13} = (m_{13}-m_{23})a_{\alpha 1} + \zeta^2(m_{33}+m_{43})$$

$$b_{23} = \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2)(m_{13}+m_{23}) + 2\rho_1\beta_1^2\zeta^2 a_{\beta 1}(m_{33}-m_{43})$$

$$b_{33} = 2a_{\alpha 1}(m_{13}-m_{23}) + (2\zeta^2-k_{\beta 1}^2)(m_{33}+m_{43})$$

equation (17)

$$A_0 = 2\,\frac{\Delta_1}{\Delta_0}$$

equation (18)

$$\underline{G}(\zeta,z,z') = \frac{2}{4\pi a_0}\sinh[a_0(h_0-z_>)] \times \cdots\cdots$$

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equation (19)

$$\underline{G}(\zeta,z,z') = \frac{2}{4\pi a_0}\sinh[a_0(h_0-z_>)]\left\{\frac{\frac{\rho_1}{\rho_0}\,a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0z_<)+a_{\alpha 1}k_{\beta 1}^4\sinh(a_0z_<)}{\frac{\rho_1}{\rho_0}\,a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0h_0)+a_{\alpha 1}k_{\beta 1}^4\sinh(a_0h_0)}\right\}$$

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